Entropy in type I algebras

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Abstract

It is shown that if (M, ϕ, α) is a W*-dynamical system with M a type I von Neumann algebra then the entropy of α w.r.t. ϕ equals the entropy of the restriction of α to the center of M. If furthermore (N, ψ, β) is a W*-dynamical system with N injective then $h_{\phi\otimes\psi}(\alpha \otimes \beta) = h_\phi(\alpha) + h_\psi(\beta)$.

1 Introduction

In the theory of non-commutative entropy the attention has almost exclusively been concentrated on non type I algebras. We shall in the present paper remedy this situation by proving the basic facts on entropy of automorphisms of type I C*- and von Neumann-algebras. The results are as nice as one can hope. The CNT-entropy of an automorphism of a von Neumann algebra of type I with respect to an invariant normal state is the classical entropy of the restriction of the automorphism to the center of the algebra. If one factor of a tensor product of two von Neumann algebras is of type I and the other injective, then the entropy of a tensor product automorphism with respect to an invariant product state is the sum of the entropies. The results have obvious corollaries to type I C*-algebras. The main idea behind our proofs is the use of conditional expectations of finite index, as employed in [GN].

We shall use the notation $h_\phi(\alpha)$ for the CNT-entropy of a C*-dynamical system as defined by Connes, Narnhofer and Thirring in [CNT], and $h'_\phi(\alpha)$ for the ST-entropy defined by Sauvageot and Thouvenot in [ST].

2 Main results

We first prove a general result for the Sauvageot-Thouvenot entropy for the restriction of an automorphism to a globally invariant C*-subalgebra of finite index. Recall the definition of ST-entropy and its connection with CNT-entropy.

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A stationary coupling of a C*-dynamical system (A,ϕ,α) with a commutative system (C,μ,β) is an $\alpha\otimes\beta$-invariant state λ on $A\otimes C$ such that $\lambda|_A=\phi$ and $\lambda|_C=\mu$. Given such a coupling and a finite-dimensional subalgebra P of C with atoms $p_1,\dots,p_n$, consider the quantity

$$H_\mu(P|P^-)-H_\mu(P)+\sum_{i=1}^n\mu(p_i)S(\phi,\phi_i),$$

where $\phi_i(a)=\frac{1}{\mu(p_i)}\lambda(a\otimes p_i)$. By definition, the ST-entropy $h'_\phi(\alpha)$ of the system (A,ϕ,α) is the supremum of these quantities.

By [ST, Proposition 4.1], ST-entropy coincides with CNT-entropy for nuclear C*-algebras. In fact, the proof of the inequality $h_\phi(\alpha)\le h'_\phi(\alpha)$ does not use any assumptions on the algebra. On the other hand, given a coupling λ and an algebra P as above, for each $m\in\mathbb{N}$ we can form the decomposition

$$\phi=\sum_{i_1,\dots,i_m=1}^n\phi_{i_1\dots i_m},\quad \phi_{i_1\dots i_m}(a)=\lambda(a\otimes p_{i_1}\beta(p_{i_2})\dots\beta^{m-1}(p_{i_m})).$$

If γ is a unital completely positive mapping of a finite-dimensional C*-algebra into A, we can use these decompositions in computing the mutual entropy $H_\phi(\gamma,\alpha\circ\gamma,\dots,\alpha^{m-1}\circ\gamma)$ [CNT]. Indeed, since the atoms in $\beta^j(P)$ are $\beta^j(p_1),\dots,\beta^j(p_n)$ we have by [CNT, III.3]

$$H_\phi(\gamma,\alpha\circ\gamma,\dots,\alpha^{m-1}\circ\gamma)\ge S\Big(\mu\Big|\bigvee_0^{m-1}\beta^j(P)\Big)-\sum_{j=0}^{m-1}S\Big(\mu\Big|\beta^j(P)\Big)$$

$$+\sum_j\sum_i\mu(\beta^j(p_i))S\Big(\phi\circ\alpha^j\circ\gamma,\frac{\lambda((\alpha^j\circ\gamma)(\cdot)\otimes\beta^j(p_i))}{\mu(\beta^j(p_i))}\Big).$$

Hence by invariance of ϕ, μ and λ with respect to α, β and $\alpha\otimes\beta$ respectively

$$\frac{1}{m}H_\phi(\gamma,\alpha\circ\gamma,\dots,\alpha^{m-1}\circ\gamma)\ge\frac{1}{m}H_\mu\Big(\bigvee_0^{m-1}\beta^j(P)\Big)-H_\mu(P)+\sum_i\mu(p_i)S(\phi\circ\gamma,\phi_i\circ\gamma).$$

It follows that

$$h_\phi(\alpha)\ge H_\mu(P|P^-)-H_\mu(P)+\sum_{i=1}^n\mu(p_i)S(\phi\circ\gamma,\phi_i\circ\gamma).$$

Thus what is really necessary for the coincidence of the entropies, is the existence of a net of unital completely positive mappings γ_i of finite-dimensional C*-algebras into A such that $S(\phi,\psi)=\lim_i S(\phi\circ\gamma_i,\psi\circ\gamma_i)$ for any positive linear functional ψ on A, $\psi\le\phi$. In particular, $h_\phi(\alpha)=h'_\phi(\alpha)$ if A is an injective von Neumann algebra and ϕ is a normal state on it.

Proposition 1 *Let (A,ϕ,α) be a unital C*-dynamical system. Let $B\subset A$ be an α-invariant C*-subalgebra (with $1\in B$). Suppose there exists a conditional expectation $E\colon A\to B$ such that $E\circ\alpha=\alpha\circ E$, $\phi\circ E=\phi$ and $E(x)\ge cx$ for all $x\in A^+$ for some $c>0$. Then $h'_\phi(\alpha)=h'_\phi(\alpha|_B)$.*

Proof. Let (C,μ,β) be a C*-dynamical system with C abelian. Using E we can lift any stationary coupling on $B\otimes C$ to a stationary coupling on $A\otimes C$. This, together with the property of monotonicity of relative entropy, shows that $h'_\phi(\alpha)\ge h'_\phi(\alpha|_B)$.

Conversely, suppose λ is a stationary coupling of (A,ϕ,α) with (C,μ,β), P a finite-dimensional subalgebra of C with atoms $p_1,\dots,p_n$, and $\phi_i(a)=\frac{1}{\mu(p_i)}\lambda(a\otimes p_i)$ for $a\in A$. Since

$\phi_i \leq \frac{1}{\mu(p_i)}\phi$, ϕ_i is normal in the GNS-representation of ϕ. Since E is ϕ-invariant, it extends to a normal conditional expectation of the closure of A in the GNS-representation onto the closure of B. Thus we can apply [OP, Theorem 5.15] to ϕ and ϕ_i, and (as in the proof of Lemma 1.5 in [GN]) get

$$\sum_{i=1}^{n} \mu(p_i) S(\phi, \phi_i) = \sum_{i=1}^{n} \mu(p_i)(S(\phi|_B, \phi_i|_B) + S(\phi_i \circ E, \phi_i)) \leq \sum_{i=1}^{n} \mu(p_i) S(\phi|_B, \phi_i|_B) - \log c.$$

It follows that $h'_\phi(\alpha) \leq h'_\phi(\alpha|_B) - \log c$. Then for each $m \in \mathbb{N}$

$$h'_\phi(\alpha) = \frac{1}{m} h'_\phi(\alpha^m) \leq \frac{1}{m} h'_\phi(\alpha^m|_B) - \frac{1}{m}\log c = h'_\phi(\alpha|_B) - \frac{1}{m}\log c.$$

Thus $h'_\phi(\alpha) \leq h'_\phi(\alpha|_B)$.

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Corollary 2 *If in the above proposition A and B are injective von Neumann algebras and ϕ is normal then $h_\phi(\alpha) = h_\phi(\alpha|_B)$.*

To prove our main result we need also two simple lemmas. The first lemma is more or less well-known.

Lemma 3 *Let (M, ϕ, α) be a W^*-dynamical system. Then*

(i) if p is an α-invariant projection in M such that $\operatorname{supp}\phi \leq p$, then $h_\phi(\alpha) = h_\phi(\alpha|_{M_p})$;

(ii) if $\{p_i\}_{i\in I}$ is a set of mutually orthogonal α-invariant central projections in M, $\sum_i p_i = 1$, then

$$h_\phi(\alpha) = \sum_i \phi(p_i) h_{\phi_i}(\alpha_i),$$

where $\phi_i = \frac{1}{\phi(p_i)}\phi$ is the normalized restriction of ϕ to Mp_i, and $\alpha_i = \alpha|_{Mp_i}$.

Proof. (i) easily follows from the definitions; (ii) follows from [CNT, VII.5(iii)], (i) and [SV, Lemma 3.3] applied to the subalgebras $M(p_{i_1} + \ldots + p_{i_n}) + \mathbb{C}(1 - p_{i_1} - \ldots - p_{i_n})$.

■

The proof of the following lemma is left to the reader.

Lemma 4 *Let T be an automorphism of a probability space (X, μ), $f \in L^\infty(X, \mu)$ a T-invariant function such that $f \geq 0$ and $\int_X f\,d\mu = 1$. Let μ_f be the measure on X such that $d\mu_f/d\mu = f$. Then $h_{\mu_f}(T) \leq ||f||_\infty h_\mu(T)$.*

Theorem 5 *Let (M, ϕ, α) be a W^*-dynamical system with M a von Neumann algebra of type I. Let Z denote the center of M. Then $h_\phi(\alpha) = h_\phi(\alpha|_Z)$.*

Proof. By Lemma 3(i) we may suppose that ϕ is faithful. Then M is a direct sum of homogeneous algebras of type I_n, $n \in \mathbb{N} \cup \{\infty\}$. By Lemma 3(ii) we may assume that M is homogeneous of type I_n. We first assume that $n \in \mathbb{N}$. Then $Z = L^\infty(X, \mu)$, where (X, μ) is a probability space and $\phi|_Z = \mu$. Thus

$$M \cong Z \otimes \mathrm{Mat}_n(\mathbb{C}) = L^\infty(X, \mathrm{Mat}_n(\mathbb{C})), \quad \phi = \int_X^\oplus \phi_x d\mu(x),$$

where $\phi_x = \mathrm{Tr}(\cdot\, Q_x)$ is a state on $\mathrm{Mat}_n(\mathbb{C})$, Tr the canonical trace on $\mathrm{Mat}_n(\mathbb{C})$. We first assume $Q_x \geq c > 0$ for all x.

If $s \in M^+$, s is a function in $L^\infty(X, \mathrm{Mat}_n(\mathbb{C}))$. Define the ϕ-preserving conditional expectation $E\colon M \to Z$ by $E(s)(x) = \phi_x(s(x))$. Then

$$E(s)(x) = \mathrm{Tr}(s(x)Q_x) \geq c\mathrm{Tr}(s(x)) \geq cs(x),$$

so $E(s) \geq cs$, and it follows from Corollary 2 that $h_\phi(\alpha) = h_\phi(\alpha|_Z)$.

If there is no $c > 0$ such that $Q_x \geq c$ for all x, let $X_c = \{x \in X \mid Q_x \geq c\}$, $(c > 0)$,

$$N_c = L^\infty(X_c, \mathrm{Mat}_n(\mathbb{C})) \ \text{ and } \ M_c = N_c + \mathbb{C}\chi_{X\setminus X_c},$$

where $\chi_{X\setminus X_c}$ is the characteristic function of $X\setminus X_c$. Since ϕ is α-invariant so is M_c, so by the above argument and Lemma 3, letting $\phi_c = \frac{1}{\mu(X_c)}\phi|_{N_c}$ and $\mu_c = \frac{1}{\mu(X_c)}\mu|_{X_c}$, we obtain

$$h_\phi(\alpha|_{M_c}) = \mu(X_c)h_{\phi_c}(\alpha|_{N_c}) = \mu(X_c)h_{\mu_c}(T|_{X_c}) \leq h_\mu(T),$$

where T is the automorphism of (X, μ) induced by α. Letting $c \to 0$ and using [SV, Lemma 3.3] we obtain the Theorem when M is finite.

If M is homogeneous of type I_∞, we have $M \cong L^\infty(X, \mu) \otimes B(H)$, where H is a separable Hilbert space. Let Tr denotes the canonical trace on $B(H)$. Write again

$$\phi = \int_X^\oplus \phi_x d\mu(x), \quad \phi_x = \mathrm{Tr}(\cdot\, Q_x),$$

and let $E_x(U)$ denote the spectral projection of Q_x corresponding to a Borel set U. Let $P_c \in M = L^\infty(X, B(H))$ be the projection defined by $P_c(x) = E_x([c, +\infty))$, where $c > 0$. Then P_c is an α-invariant finite projection. Let

$$M_c = P_cMP_c + \mathbb{C}(1 - P_c).$$

Then M_c is a finite type I von Neumann algebra. Its center is isomorphic to $L^\infty(X_c, \mu_c) \oplus \mathbb{C}$, and the restriction of ϕ to it is $\phi(P_c)\mu_c \oplus \phi(1 - P_c)$, where $X_c = \{x \in X \mid P_c(x) \neq 0\}$ and

$$\int_{X_c} f(x)d\mu_c(x) = \frac{1}{\phi(P_c)}\int_{X_c} f(x)\phi_x(P_c(x))d\mu(x).$$

So we can apply the first part of the proof to M_c. Since $d\mu_c/d\mu \leq \frac{1}{\phi(P_c)}$, applying Lemma 4 we get

$$h_\phi(\alpha|_{M_c}) = \phi(P_c)h_{\mu_c}(T|_{X_c}) \leq h_\mu(T).$$

Now letting $c \to 0$ we conclude that $h_\phi(\alpha) = h_\mu(T)$.

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It should be remarked that in a special case the above theorem was proved in [GS, Proposition 2.4].

If A is a C*-algebra and ϕ a state on A, the central measure μ_ϕ of ϕ is the measure on the spectrum $\hat{A}$ of A defined by $\mu_\phi(F) = \phi(\chi_F)$, where ϕ is regarded as a normal state on A'', see [P, 4.7.5]. Thus by Theorem 5 and [P, 4.7.6] we have the following

Corollary 6 *Let (A, ϕ, α) be a C*-dynamical system with A a separable unital type I C*-algebra. Then $h_\phi(\alpha) = h_{\mu_\phi}(\hat{\alpha})$, where $\hat{\alpha}$ is the automorphism of the measure space $(\hat{A}, \mu_\phi)$ induced by α.*

Since inner automorphisms act trivially on the center we have

Corollary 7 *If (M, ϕ, α) is a W*-dynamical system with M of type I and α an inner automorphism then $h_\phi(\alpha) = 0$.*

Note that in the finite case the above corollary also follows from a result of N. Brown [Br, Lemma 2.2].

The next result was shown in [S] when ϕ is a trace.

Corollary 8 *Let R denote the hyperfinite II_1-factor. Let A be a Cartan subalgebra of R and u a unitary operator in A. If ϕ is a normal state such that u belongs to the centralizer of ϕ then $h_\phi(\mathrm{Ad}\, u) = 0$.*

Proof. As in [S], it follows from [CFW] that there exists an increasing sequence of full matrix algebras $N_1 \subset N_2 \subset \dots$ with union weakly dense in R such that $A \cong A_n \otimes B_n$, where $A_n = N_n \cap A$ and $B_n = (N_n' \cap R) \cap A$ for all $n \in \mathbb{N}$. Let $M_n = N_n \otimes B_n$. Then M_n is of type I and contains u. Hence $h_\phi(\mathrm{Ad}\, u|_{M_n}) = 0$. Since $(\cup_n M_n)^- = R$, $h_\phi(\mathrm{Ad}\, u) = 0$ by [SV, Lemma 3.3].

■

If (A, ϕ, α) and (B, ψ, β) are C*-dynamical systems we always have

$$h_{\phi\otimes\psi}(\alpha \otimes \beta) \geq h_\phi(\alpha) + h_\psi(\beta),$$

see [SV, Lemma 3.4]. The equality does not always hold, see [NST] or [Sa]. However, we have

Theorem 9 *Let (A, ϕ, α) and (B, ψ, β) be W*-dynamical systems. Suppose that A is of type I, and B is injective. Then*

$$h_{\phi\otimes\psi}(\alpha \otimes \beta) = h_\phi(\alpha) + h_\psi(\beta).$$

Proof. We shall rather prove that $h_{\phi\otimes\psi}(\alpha \otimes \beta) = h_\phi(\alpha|_{Z(A)}) + h_\psi(\beta)$. For this it suffices to consider the case when A is abelian; the general case will follow by the same arguments as in the proof of Theorem 5. (Note that the mapping $x \mapsto \mathrm{Tr}(x) - x$ on $\mathrm{Mat}_n(\mathbb{C})$ is not completely positive, but the mapping $x \mapsto \mathrm{Tr}(x) - \frac{1}{n}x$ is by the Pimsner-Popa inequality. Thus replacing M with $M \otimes B$ and Z with $Z \otimes B$ in the proof of Theorem 5 we have to replace the inequality $E(s) \geq cs$ in the proof with $E(s) \geq \frac{c}{n}s$.)

So suppose that A is abelian. It is clear that it suffices to prove that if $A_1, \dots, A_n$ are finite-dimensional subalgebras of A, and $B_1, \dots, B_n$ are finite-dimensional subalgebras of B, then

$$H_{\phi\otimes\psi}(A_1 \otimes B_1, \dots, A_n \otimes B_n) = H_\phi(A_1, \dots, A_n) + H_\psi(B_1, \dots, B_n).$$

We always have the inequality "$\geq$", [SV, Lemma 3.4]. To prove the opposite inequality consider a decomposition

$$\phi \otimes \psi = \sum_{i_1,\dots,i_n} \omega_{i_1\dots i_n}.$$

Let $H_{\{\phi\otimes\psi=\sum \omega_{i_1\dots i_n}\}}(A_1 \otimes B_1, \dots, A_n \otimes B_n)$ be the entropy of the corresponding abelian model, so

$H_{\{\phi\otimes\psi=\sum \omega_{i_1\dots i_n}\}}(A_1 \otimes B_1, \dots, A_n \otimes B_n) =$

$$= \sum_{i_1,\dots,i_n} \eta\omega_{i_1\dots i_n}(1) + \sum_{k=1}^{n} \sum_{i} S\left(\phi \otimes \psi|_{A_k\otimes B_k}, \sum_{i_k=i} \omega_{i_1\dots i_n}|_{A_k\otimes B_k}\right).$$

Set $C = \vee_{k=1}^n A_k$. Let $p_1, \dots, p_r$ be those atoms p of C for which $\phi(p) > 0$. Define positive linear functionals $\psi_{m,i_1\dots i_n}$ on B,

$$\psi_{m,i_1\dots i_n}(b) = \frac{\omega_{i_1\dots i_n}(p_m \otimes b)}{\phi(p_m)}.$$

Let also ϕ_m be the linear functional on C defined by the equality $\phi_m(a) = \phi(ap_m)$. Then

$$\omega_{i_1\dots i_n} = \sum_{m=1}^r \phi_m \otimes \psi_{m,i_1\dots i_n} \quad \text{on} \quad C \otimes B,$$

and

$$\psi = \sum_{i_1,\dots,i_n} \psi_{m,i_1\dots i_n} \quad \text{for} \quad m = 1, \dots, r.$$

Since the supports of the states ϕ_m are mutually orthogonal minimal projections in C, we have

$$\sum_{k=1}^n \sum_i S\left(\phi \otimes \psi|_{A_k \otimes B_k}, \sum_{i_k=i} \omega_{i_1\dots i_n}|_{A_k\otimes B_k}\right) \leq$$

$$\begin{aligned}
&\leq \sum_{k=1}^n \sum_i S\left(\phi \otimes \psi|_{C\otimes B_k}, \sum_{i_k=i} \omega_{i_1\dots i_n}|_{C\otimes B_k}\right) \\
&= \sum_{k=1}^n \sum_i S\left(\phi \otimes \psi|_{C\otimes B_k}, \sum_{m=1}^r \phi_m \otimes \left(\sum_{i_k=i} \psi_{m,i_1\dots i_n}\right)|_{C\otimes B_k}\right) \\
&= \sum_{k=1}^n \sum_i \sum_{m=1}^r \phi(p_m) S\left(\psi|_{B_k}, \sum_{i_k=i} \psi_{m,i_1\dots i_n}|_{B_k}\right).
\end{aligned}$$

If $a_i \geq 0$ and $\sum\limits_i a_i \leq 1$ then $\eta(\sum\limits_i a_i) \leq \sum\limits_i \eta(a_i)$. Hence we have

$$\begin{aligned}
\sum_{i_1,\dots,i_n} \eta\omega_{i_1\dots i_n}(1) &\leq \sum_{m=1}^r \sum_{i_1,\dots,i_n} \eta(\phi_m \otimes \psi_{m,i_1\dots i_n})(1) \\
&= \sum_{m=1}^r \eta\phi(p_m) \sum_{i_1,\dots,i_n} \psi_{m,i_1\dots i_n}(1) + \sum_{m=1}^r \phi(p_m) \sum_{i_1,\dots,i_n} \eta\psi_{m,i_1\dots i_n}(1) \\
&= \sum_{m=1}^r \eta\phi(p_m) + \sum_{m=1}^r \phi(p_m) \sum_{i_1,\dots,i_n} \eta\psi_{m,i_1\dots i_n}(1).
\end{aligned}$$

Thus

$$\begin{aligned}
&H_{\{\phi\otimes\psi=\sum \omega_{i_1\dots i_n}\}}(A_1 \otimes B_1, \dots, A_n \otimes B_n) \leq \\
&\quad \leq \sum_{m=1}^r \eta\phi(p_m) + \sum_{m=1}^r \phi(p_m) H_{\{\psi=\sum \psi_{m,i_1\dots i_n}\}}(B_1, \dots, B_n).
\end{aligned}$$

Since $\sum_m \eta\phi(p_m) = H_\phi(C) = H_\phi(A_1, \dots, A_n)$, we conclude that

$$H_{\phi\otimes\psi}(A_1 \otimes B_1, \dots, A_n \otimes B_n) \leq H_\phi(A_1, \dots, A_n) + H_\psi(B_1, \dots, B_n),$$

completing the proof of the Theorem.

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